Some Exact Solutions of the Navier-Stokes Equations

We next turn our attention to the exact solutions of incompressible Navier-Stokes equations. Even though these equations were derived over a century ago, only a handful of exact solutions exist for some highly simplified situations. This is because of the nonlinear nature of these equations.

The governing equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\rho\frac{Du}{Dt}+\frac{\partial p}{\partial x}=\mu\nabla^2 u$$

$$\rho\frac{Dv}{Dt}+\frac{\partial p}{\partial y}=\mu\nabla^2 v$$

$$\rho\frac{De}{Dt}+\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}=\frac{\partial\left(u\tau_{xx}+v\tau_{xy}\right)}{\partial x}+\frac{\partial\left(u\tau_{xy}+v\tau_{yy}\right)}{\partial y}+k\nabla^2 T$$

where,

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

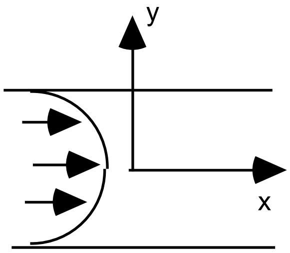
$$e=C_v T+\frac{u^2+v^2}{2}$$

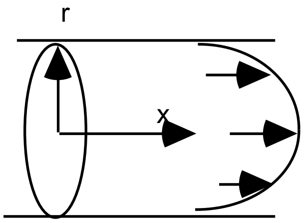
In the above equations, we have assumed the conductivity k and viscosity □ to be constant. We will further restrict ourselves to steady flows ($\partial/\partial t = 0$) , and incompressible flows (□ = constant).

The above form of equations apply only to 2-D planar flows. Similar forms of Navier-Stokes equations exist in other coordinate systems such as the cylindrical, polar and spherical coordinate systems.

Parallel Flows

The first class of flows to be considered is the flow between (or within) infinitely long parallel plates, and infinitely long tubes.





In such flows, it is reasonable to assume that the flow behavior will be independent of the x- location. That is, x-derivatives of all flow properties such as $\partial u/\partial x$, $\partial v/\partial x$, $\partial T/\partial x$ etc. vanish. The pressure derivative $\partial p/\partial x$ is assumed to be a constant, required to drive the flow.

In such as situation, continuity equation becomes

Cartesian Form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Cylindrical Coordinates:

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r}=0$$

Setting $\partial u/\partial x=0$ yields $\partial v/\partial y = 0$ or $\partial(rv)/\partial r=0$. The assumption $\partial v/\partial x=0$ implies that the velocity component v is not a function of x.

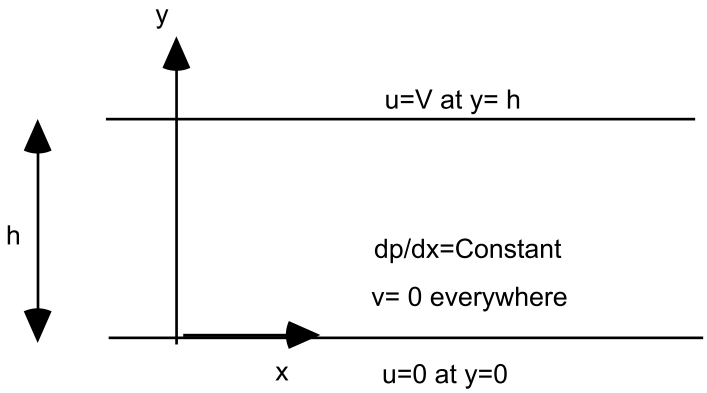
Thus, continuity yields v = constant, for our flows. Since the boundary condition requires v to be zero at the walls of our parallel plates, and at the walls of the tube, continuity + boundary conditions together yield

$$v = 0$$

In other words, the flow only has a u- component and is parallel to the x-axis. For this reason, the first class of flows we study are called parallel flows.

Exact Solution # 1: Planar Couette Flow:

In 1890 Couette performed experimental studies of flow between two concentric cylinders, where one of the cylinder is fixed, the other is spinning. This situation is similar to what happens in a ball bearing. here we study the 2-D analog of flow between two parallel plates. One of the plate is held fixed, while the other one is moving at a velocity V. The plates are separated by a distance h. A constant pressure gradient dp/dx is applied to this flow.



For this case, the u- momentum equation yields

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

where,

$$\nu=\frac{\mu}{\rho}=\text{Kinematic Viscosity}$$

Setting the x- derivatives to zero, and requiring v to ve zero, the above equation may be simplified to yield

$$\frac{1}{\rho}\frac{dp}{dx}=\nu\frac{d^2 u}{dy^2}$$

Notice that we have begun to replace partial derivatives with respect to y with ordinary derivatives, because flow properties are only functions of y.

Integrating the above equation twice, we get:

$$u(y)=\frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2}+Ay+B$$

where A and B are constants of integration. These may be evaluated by requiring u=0 at y=0 and u= U at y=h.

$$u(y) = U\frac{y}{h} + \frac{1}{\mu}\frac{dp}{dx}\frac{y}{2}(y-h)$$

Two special situations are of interest. In the first case, the pressure gradient dp/dx=0 is zero, and the flow motion is brought about by the motion of the top plate at the constant velocity V. In that case, the velocity profile is linear, giving

$$u(y) = U\, y/h$$

In the second situation, both the plates are at rest and V=0. the fluid is motion is caused by the application of pressure gradient dp/dx. In this case, the velocity profile is a parabola, given by

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y(y-h)$$

The shear stress τ_{xy}(i.e. the force per unit area exerted by the fluid on the flat plates, and vice versa) may be computed from Stokes' relations. For example, for the case where dp/dx=0, we get

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{U}{h}$$

The non-dimensional "skin friction" coefficient C_f is then computed as

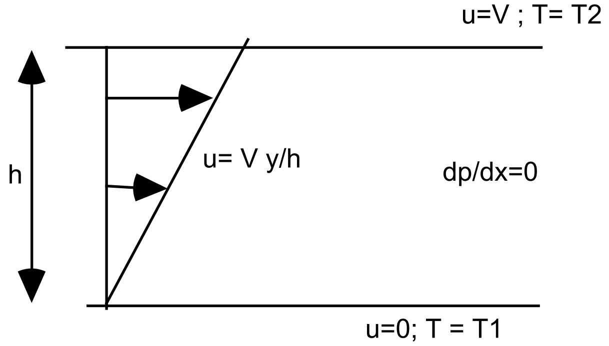
$$C_f = \frac{\tau_{xy}}{\frac{1}{2}\rho U^2} = \frac{2\mu}{\rho h U} = \frac{2}{\mathrm{Re}}$$

where,

$$\mathrm{Re} = \text{Reynolds number based on U and h} = \frac{\rho U h}{\mu}$$

<u>Temperature Distribution for Planar Couette Flow:</u>

The energy equation, which is part of the Navier-Stokes equations is usually not solved in incompressible flow applications, unless we are interested in a heat transfer application. Let us consider the situation where dp/dx=0, the top plate is moving at the velocity U, and the pressure gradient dp/dx=0. Let the bottom and top plates be at two different temperatures T_1 and T_2 respectively. In this case, we can easily solve for the temperature distribution in the fluid between the plates, and compute the rate at which heat is transferred from the hot plate to the cold plate, as follows.



The energy equation for this case is

$$\frac{\partial(\rho u h_0)}{\partial x}+\frac{\partial(\rho v h_0)}{\partial y}=\frac{\partial\left(u\tau_{xx}+v\tau_{xy}\right)}{\partial x}+\frac{\partial\left(u\tau_{xy}+v\tau_{yy}\right)}{\partial y}+k\nabla^2 T$$

where,

$$h_0=\text{Stagnation enthalpy}=\mathrm{C_v}T+\frac{p}{\rho}+\frac{u^2+v^2}{2}$$

For our parallel flow, we can set ∂/∂x =0 and v = 0. For the case where dp/dx=0, the velocity profile is linear, as derived earlier. Then,

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\tau_{yy}=2\mu\frac{\partial v}{\partial y}+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\mu\frac{U}{h}$$

When these expressions are substituted into the energy equation, and when all x-derivatives are set to zero, and when v is set to zero, the following form results.

$$k\frac{\partial^2 T}{\partial y^2} = -\mu\frac{U^2}{h^2}$$

Integrating the above equation twice, we get

$$T(y) = -\frac{\mu}{2k}\frac{U^2}{h^2}y^2 + Cy + D$$

where C and D are constants of integration. These constants may be found by applying the boundary condition T= T_1 at y=0, and T = T_2 at y= h. The final form is

$$T(y) = T_1 + \frac{T_2 - T_1}{h}y + \frac{\mu}{2k}U^2\left[\frac{y}{h} - \frac{y^2}{h^2}\right]$$

The first two terms in the above expression are linear in y, and model the effect of conduction on the temperature distribution. The third term models the effect of heat geeration due to viscous work. Since □ is small, this term is likely to be of significance only in high speed flows.