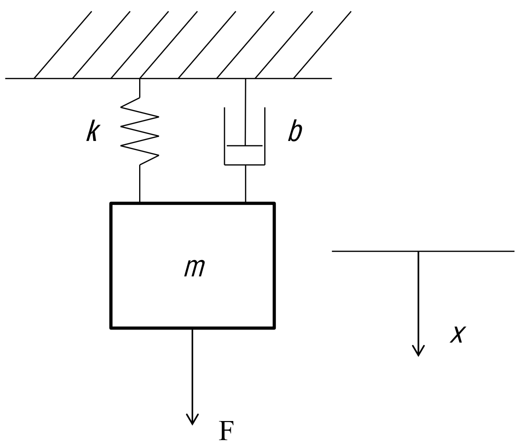
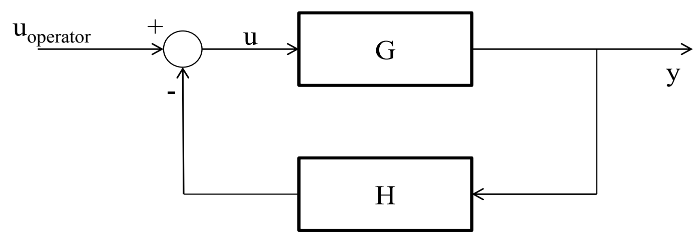
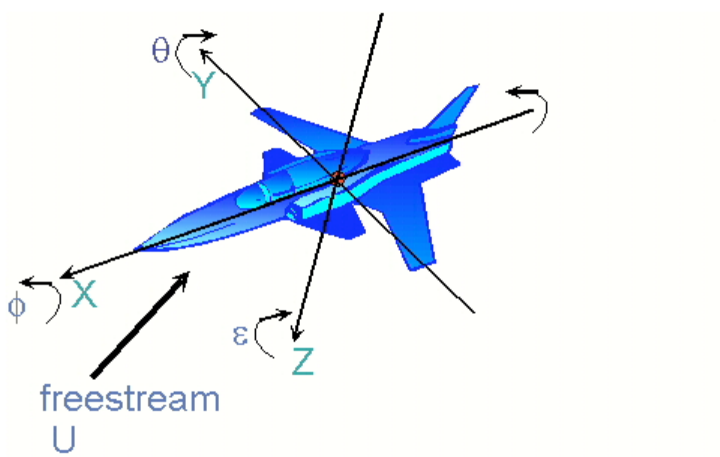
Extrovert E-book Series

System Dynamics

Louis Dressel







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Georgia Institute of Technology

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Contents

Chapter 1

Math Review

1.1 Complex Number Review

Polar Form

$$z = |z|e^{i\theta} \tag{1.1}$$

nth Root of a Complex Number

$$z^{1/n} = |z|^{1/n} e^{i(\theta + 2\pi k)/n} \tag{1.2}$$

1.2 Zeros/Poles of a Function

The **zeros** of a function $f(x)$ are all values of x such that $f(x) = 0$. These values of x make the numerator equal to zero.

The **poles** of a function $f(x)$ are all the values of x for which $f(x)$ approaches infinite. These values of x make the denominator equal to zero.

1.3 Linearization

Linear functions are much easier to work with than nonlinear functions. Unfortunately, not everything worth modeling acts in a linear fashion.

For example, consider the equation for the drag force on an object.

$$D = \frac{1}{2}\rho v^2 S C_D$$

Assuming a constant density, reference area, and drag coefficient, drag only varies as a function of the square of velocity. This is not linear relationship.

The solution to this is to linearize the function in question. This requires taking a line tangent to the function at some point and using the line to represent the function. This is shown in Figure 1.1. In this example, the function y is a function of x. The value of x at which the tangent line is generated is denoted by $\bar{x}$. The linear approximation is only valid for a limited range of x-values around $\bar{x}$. If the function is highly non-linear, this range could be very small.

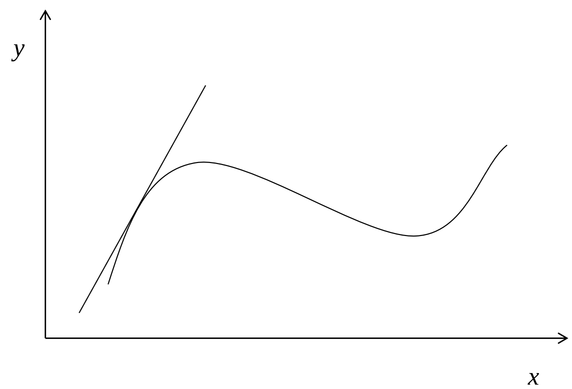

Figure 1.1: Example of linearization.

The process of linearizing a function $f(x)$ about some value $\bar{x}$ involves finding the function value at $\bar{x}$, and finding the slope of the tangent line at this point (with calculus). The values of points near $\bar{x}$ can be determined from this. The process is shown below:

$$f(x) \simeq f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x - \bar{x})$$

A multivariable function $z = f(x, y)$ can be linearized about $\bar{x}$ and $\bar{y}$:

$$z \simeq f(\bar{x}, \bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ y=\bar{y}}} (x - \bar{x}) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=\bar{x} \\ y=\bar{y}}} (y - \bar{y})$$

Chapter 2

The Laplace Transform

The Laplace transform is a type of transform function. Transform functions maps a function $f(t)$ in one domain into another function $F(s)$ in another domain. The general equation of a transform function is shown in Equation 2.1:

$$F(s) = \int_{\alpha}^{\beta} k(t,s)f(t)dt \tag{2.1}$$

In Equation 2.1, $F(s)$ is the transform of $f(t)$. The function $k(t,s)$ is the kernel of the transform. In the Laplace transform, the kernel is $k(t,s) = e^{-st}$. The Laplace transform is shown below.

$$F(s) = L[f(t)] = \int_{0}^{\infty} e^{-st}f(t)dt \tag{2.2}$$

2.1 Properties of the Laplace Transform

The Laplace transform is a linear operator. This means it comforms to the two defining properties of linearity: homogeneity and additivity.

Homogeneity is also called the scalar rule. Let the constant a be some real or complex number:

$$L(af(t)) = aL(f(t))$$

Additivity is also called the superposition principle. The Laplace transform of the sum of two functions f_1 and f_2 is equal to the sum of the Laplace transforms of the individual functions.

$$L(f_1(t) + f_2(t)) = L(f_1(t)) + L(f_2(t))$$

Combined, these properties mean the Laplace transform is a linear operator.

$$L(a_1f_1(t) + a_2f_2(t)) = a_1L(f_1(t) + a_2L(f_2(t))$$

Differentiation

$$L[f'(t)] = sL(f(t)) - f(0)$$
$$L[f''(t)] = s^2L(f(t)) - sf(0) - f'(0)$$
$$L(f^{(n)}(t)) = s^nL(f(t)) - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{2.3}$$

Integration

Final Value Theorem

If $\lim_{t\to\infty} f(t)$ exisits, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Initial Value Theorem

If the Laplace transform of $f(t)$ and $\frac{d}{dt}f(t)$ exists, and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

It is important to note that $f(0) = \lim_{s\to\infty} sF(s)$ is usually true too. However, it is not true for impulse functions.

Chapter 3

Systems and System Types

3.1 Definition of Systems

3.2 Input Range

A **static system** is a system in which the output is a function of the current input.

A **dynamic system** is a system in which the output is a function of the current input and its history.

A **causal system** is a system in which the output does not depend on future input. Most systems are causal.

3.3 Linearity

In a **linear system**, the governing equation is linear.

In a **nonlinear system**, the governing equation is nonlinear.

3.4 Time Variance

Chapter 4

Mechanical Systems

4.1 Elements of Mechanical Systems

There are three basic elements of mechanical systems that we will consider. Each element exerts a force to resist a different kinematic quantity. Spring elements resist displacement, damper elements resist velocity, and mass elements resisit acceleration. The general equation for the force exerted by a basic mechanical element is:

$$\text{Force} = -(\text{constant})(\text{kinematic quantity})$$

The negative sign is present because the force always resists the kinematic quantity, and thus works against it.

Spring Element

Spring elements resist displacement. The force F exerted by a spring is given by Hooke's law:

$$F = -kx$$

where k is known as the spring constant, and x is the displacement.
When springs are in parallel, the total spring constant can be found by adding the two individual spring constants.
When springs are in in series, their spring constant can be found by adding the reciprocals of individual spring constants, and taking the reciprocal of this value.

Damper Element

Damper elements resist velocity. If an object connected to a damper is in motion, a damper will exert a force to stop the motion. The damping force exerted in response to velocity can be found with the following relationship:

$$F = -b\dot{x}$$

As shown in the above equation, we denote the damper constant with b, and $\dot{x}$ is the velocity.

Mass Element

Mass elements resisit acceleration.

$$F = -m\ddot{x}$$

The negative sign makes this different from the $F = ma$ you might already be familiar with. However, the F in that equation is referring to the total force exerted on the object, which acts in the direction of acceleration. In our equation, we are discussing the inertial resistance to acceleration, which acts opposite to the acceleration. This is the reason for the negative sign.
This difference can also be explained with d'Alembert's Principle. D'Alembert's principle is simply an alternative form of Newton's second law. While Newton's second law can be expressed

$$\vec{F} = m\vec{a}$$

D'Alembert's principle expresses it as

$$\vec{F} - m\vec{a} = 0$$

This allows a problem in dynamics to be reduced to a problem in statics. This latter format suggests an equilibrium between the force $\vec{F}$ exerted on the object and the inertial force $-m\vec{a}$ that opposes the object.

4.2 Basic Mechanical Systems

Mass-Spring System

This system includes a mass element and a spring element. This system is shown in Figure 4.1.

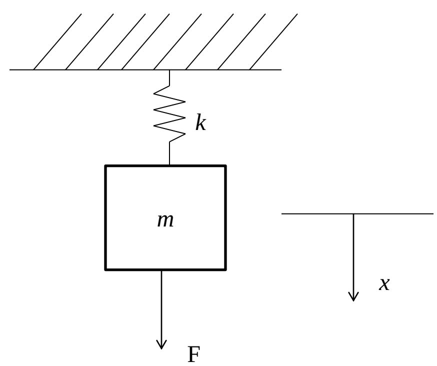

Figure 4.1: Mass-spring system experiencing a force.

The coordinate system defines positive x as downward.

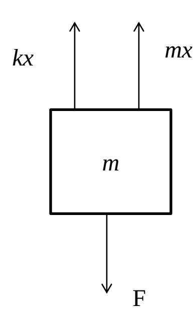

Figure 4.2: Force diagram of mass-spring system.

The mathematical model of the system is shown below:

$$m\ddot{x} + kx = F$$

Mass-Spring-Damper System

This system includes a mass element, a spring element, and a damper element.

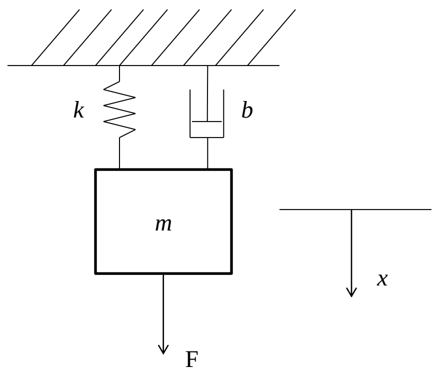

Figure 4.3: Mass-spring-damper system.

The mathematical model of the mass-spring-damper system is shown below.

$$m\ddot{x} + b\dot{x} + kx = F$$

4.3 Degrees of Freedom

Chapter 5

Transfer Functions

5.1 Transfer Function Definition

The **transfer function** of a dynamic system is the ratio between the Laplace transform of the output over the Laplace transform of the input with all initial conditions set to zero.

$$\text{Transfer Function} = \frac{\text{L(output)}}{\text{L(input)}}$$

We denote the transfer function with G, the Laplace transform of the output with Y, and the Laplace transform of the input with U or X:

$$G(s) = \frac{Y(s)}{U(s)} \tag{5.1}$$

Because Laplace transforms convert differential equations into algebraic equations, the transfer function is always a ratio of polynomials.

Example:

5.2 Block Diagrams

5.3 Convolution Integral

Given $g(t)$, the convolution integral can be used to find the response to some input $u(t)$.

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau \tag{5.2}$$

This is especially useful in finding the unit step response.

Chapter 6

State-Space Models

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Matrix A is the state matirx.
Matrix B is the input matrix.
Matrix C is the output matrix.
Matrix D is the direct transmission matrix.

Chapter 7

Time-Domain Analysis of Dynamic Systems

7.1 Dynamic System Responses

The response, or solution, of a system refers to
For example, recall the mathematical model of a mass-spring-damper system:

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{7.1}$$

The solution, or response, to this system would simply be the position as a function of time, $x(t)$.

$$x(t) = x_c(t) + x_p(t)$$

Complementary Solution

The complementary solution assumes no input. In equation 7.1, this would be achieved if $f(t)$ were simply set to zero. Because it only relies on the initial conditions, and not the input, the complementary solution is often called the initial condition response. It is also called the natural response.

Particular Solution

The particular solution is often called the forced response.

Transient Response

7.2 First-Order Systems

The general equation of a first order system is:

$$\dot{x} + ax = bu$$

Where a and b are simply constants. The transfer function G(s) can be found using the Laplace transform:

$$G = \frac{X}{U} = \frac{b}{s+a}$$

It becomes useful to define the transfer function in terms of a time constant τ and a gain k.

$$\tau\dot{x} + x = ku \tag{7.2}$$

Initial Condition Response

The complementary, or initial condition, response, is the response of the system to specific initial conditions and no input. Thus, in solving for it, the right-hand side of Equation 7.2 is set to zero.

$$\tau \dot{x} + x = 0$$

Taking the Laplace transform of this function yields

$$\tau s X - \tau x_0 + X = 0$$

Rearranging

$$X = \frac{\tau x_0}{\tau s + 1}$$

Taking the inverse Laplace yields

$$x(t) = x_0 e^{-t/\tau} \tag{7.3}$$

Unit Impulse Response

If $u = \delta(t)$, what is $x(t)$? Because the Laplace transform of the unit impulse function is one, the output of the system $x(t)$ is equal to the inverse Laplace of the transfer function.

$$\begin{aligned} x(t) &= g(t) = L^{-1}(G(s)) \\ &= \frac{k}{\tau} e^{-t/\tau} \end{aligned} \tag{7.4}$$

Unit Step Response

If $u = 1(t)$, what is $x(t)$? The convolution integral can be used for this problem.

$$\begin{aligned} x(t) &= \int_0^t g(t_1) u(t - t_1) dt_1 \\ &= \int_0^t \frac{k}{\tau} e^{-t_1/\tau} dt_1 \\ &= \frac{k}{\tau} [-\tau e^{-t_1/\tau}] \Big|_0^t \\ &= k[1 - e^{-t/\tau}] \end{aligned} \tag{7.5}$$

7.3 Second-Order Systems

A typical second-order system is the mass-spring-damper system that has already been explored.

$$m\ddot{x} + b\dot{x} + kx = u$$

Imagine that there is no damping in this system ($b = 0$). If displaced from its equilibrium position, the mass would oscillate indefinitely. This would not happen in reality, because no real system is completely devoid of damping. But an ideal second-order system with no damping would oscillate forever. This is shown in Figure 7.1. The amplitude of the response is constant because no damper removes energy from the system.

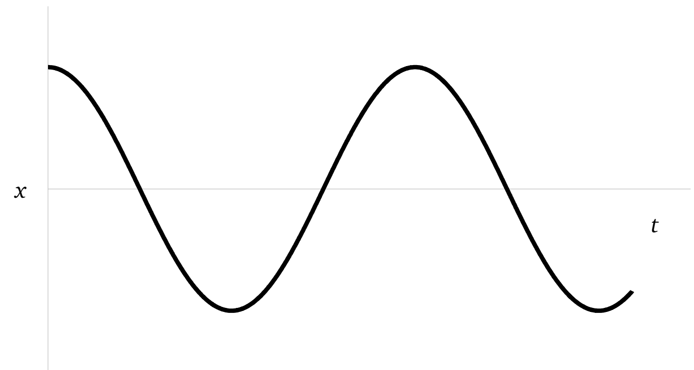

Figure 7.1: Initial condition response of a second-order system with no damping.

The frequency at which the undamped system oscillates is known as the **natural frequency**. For any second-order system, damped or not, the natural frequency is the square root of the spring constant divided by the mass. Equation 7.6 shows how the natural frequency, ω_n, is calculated. If the units are consistent, this frequency is given in rad/s.

$$\omega_n = \sqrt{\frac{k}{m}} \tag{7.6}$$

Now imagine that there is light damping, so that the damping constant is now greater than zero. Because the damping component reduces the energy of the system, the amplitude of the response continually diminishes until it is zero. The oscillations no longer occur at the natural frequency, but occur at a lower frequency. If the damping is increased, the amplitudes diminish more quickly, and the frequency of the oscillation becomes smaller.

Figure 7.2: Effect of slight damping on the response of a second-order system.

If the damping is increased to a certain value, the system no longer oscillates, and converges to the steady state value. At this value of the damping constant, the system is said to be critically damped. For damping constants less than this value, the system is under-damped. For damping constants larger than this value, the system is over-damped. The ratio of the damping constant to the critical damping constant is known as the **damping ratio**, denoted by ζ. Equation 7.7 shows the how the damping ratio is calculated from the mass, spring constant, and damping constant.

$$\zeta = \frac{b}{2\sqrt{km}} \tag{7.7}$$

It has been noted that damping causes a system to oscillate at a frequency other than its natural frequency. The frequency at which the oscillations occur in a damped system is the **damped frequency**, ω_d. If the damping is increased on the system, the damped frequency decreases.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{7.8}$$

Now that the concepts of natural frequency and damping ratio have been introduced, it becomes convenient to rewrite the mathematical model of the second-order system in these terms:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u.$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}u \tag{7.9}$$

7.4 Goals

1. What are the complementary and particular solutions of a dynamic system?
2. What effect does damping have on the response of a system?
3. Define the natural frequency, damped frequency, and damping ratio of a second-order system.

Chapter 8

Frequency-Domain Analysis of Dynamic Systems

Chapter 9

Basic Control Theory

Until this point, we have only explored the responses of systems to specific inputs. However, it is useful to modify a system such that it responds to specific inputs in a particular manner. In order to control an existing system, or plant, a controller is added. This controller modifies the system so that it will respond to inputs in a manner that we designate. A successfully controlled system will produce correct outputs based on the input of a user.

Control systems are divided into two main groups depending on their use of **feedback**. Feedback is the comparison of system output to the desired input. Systems that use feedback are known as closed-loop systems. Systems that do not use feedback are known as open-loop systems.

9.1 Closed-Loop System

Closed-loop systems are systems that employ the use of feedback. They are commonly called feedback control systems.

9.2 Open-Loop System

9.3 Block Diagrams

Including Feedback

Mason's Rule for Block Diagram Reduction

$$G = \frac{\text{Sum of forward path gains}}{1 + \text{sum of loop path gains}} \tag{9.1}$$

9.4 SAS vs CAS

As shown in the figure above, the controller can be in the forward path or in the feedback path. Positioning the controller in the feedback path yields a stability augmentation system (SAS), and positioning the controller in the forward path yields a command augmentation system (CAS).

Stability Augmentation System

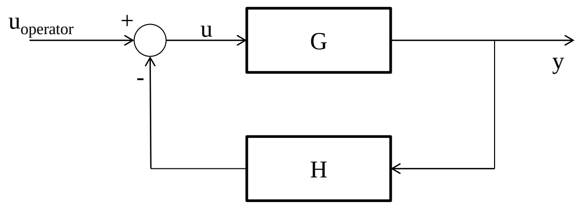

Figure 9.1: Stability Augmentation System.

Command Augmentation System

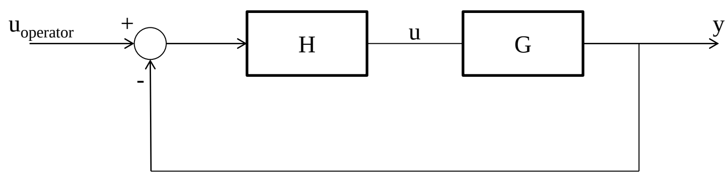

Figure 9.2: Command Augmentation System.

Chapter 10

Control Systems

10.1 P-Controller

$$C = Kp \tag{10.1}$$

10.2 PI-Controller

$$C = K_p + \frac{K_I}{s} \tag{10.2}$$

$$C = \frac{K_I(\frac{K_p}{K_I}s + 1)}{s} \tag{10.3}$$

10.3 PD-Controller

$$C = K_p + sK_d \tag{10.4}$$

10.4 PID-Controller

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