Welcome to
High Speed Aerodynamics
WHAT IS HIGH SPEED AERODYNAMICS?

Lift, drag and pitching moment?
- Linearized Potential Flow
- Transformations
- Compressible Boundary Layer

Airfoil section?
- Thin airfoil theory
- Critical Mach number
- Shock-expansion theory

Body Shape for best volume and lowest drag?
- Slender Body Theory

Wing Planform?
- Sweep
- Slender Wing

Configuration?
- Supersonic Area Rule
- Wing/Body/Nacelle Interference

Nose shape?
- Conical Flow
- Karman theory

http://www.aviationexplorer.com/Supersonic%20Aircraft/japan_supersonic.jpg
1. Review of Low Speed Aerodynamics and Gas Dynamics
Basic Concepts and Results in Aerodynamics

**Freestream Vector**: Velocity of the fluid far ahead of the object in the flow, undisturbed by the presence of the object.

**Aerodynamic Lift**: Force perpendicular to the freestream, exerted by the flow on an object.

\[ L = q_\infty S C_L \]

**Drag**: Force along the freestream, acting on the aircraft.

\[ D = q_\infty S C_D \]

**Dynamic Pressure in low-speed flow**: \[ q_\infty = \frac{1}{2} \rho U_\infty^2 \]
Lift-Curve Slope of an airfoil:

\[
\frac{dc_l}{d\alpha} \leq 2\pi
\]

This is derived from "thin-airfoil theory" in low-speed flows where Mach number is close to 0.
Effect of Mach Number on Lift Curve Slope

In subsonic flow,
\[
\frac{dC_1}{d\alpha} \bigg|_{M} = \frac{dC_1}{d\alpha \bigg|_{M=0}} \frac{1}{\sqrt{1 - M^2}}
\]

In supersonic flows,
\[
\frac{dC_1}{d\alpha} \bigg|_{M} = \frac{4}{\sqrt{M^2 - 1}}
\]
Kutta-Jowkowsky Theorem

Derived from Newton’s 2nd Law.
Derived by observing that there are 2 ways that lift can be explained:

1. Due to circulation (flow turning)

   Lift per unit span \( \bar{L}' = \rho U_\infty \times \bar{\Gamma} \)

   Here \( \bar{\Gamma} \) is the Bound Circulation

2. Due to compression / expansion of the flow (density changes due to velocity changes):
   this cannot happen in incompressible flow, but we will see this type in this course.
Finite Wing Vortex system

Aspect Ratio: $AR = \frac{b^2}{S}$ where $b$ is the span and $S$ is the planform area of the wing.

For an (2-D) airfoil $AR = \infty$

Spanwise Load Distribution

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Pressure difference between upper and lower surfaces must vanish at tips of the wing. There are two effects of this,

a) Loss in lift, compared to 2-D airfoils.

b) Induced drag, i.e. drag induced by the force vector tilting backwards as a result of the induced downwash.
Points to Note on Finite Wings

1. Since induced drag is directly related to the lift it can be calculated by the same mathematical formulation used to calculate lift.
2. Does not require consideration of viscosity.
3. No induced drag on 2-D airfoils under steady conditions.
4. At finite Aspect ratios,

\[ C_{Di} = \frac{C_L^2}{\pi (AR)e} \]

5. Ideal elliptic lift distribution implies minimum induced drag, i.e,

spanwise efficiency e = 1.

6. At low Mach numbers drag on well designed aircraft is primarily due to induced drag.

7. Note: In the 2-D limit (airfoil) there is no lift-induced drag in incompressible flow. There is always “profile” drag due to viscous effects (friction drag) and flow separation (pressure drag).

8. “Potential Flow” (inviscid) analysis cannot capture the profile drag in subsonic flow. But it can capture the lift-induced drag on finite wings.
Effect of Aspect Ratio on Wing Lift Curve Slope

\[ \frac{dC_L}{d\alpha} \equiv a = \frac{a_0}{1 + \left( \frac{a_0}{\pi(AR)} \right)(1 + \tau)} \]

\[ (1 + \tau) \approx \frac{1}{e} \]

\[ a_0 \equiv \frac{dc_l}{d\alpha} \]
Check your understanding

1. If the freestream speed is 10 m/s, the density is 1.2kg/m³ and the circulation is 2m²/s, what is the lift per unit span?
2. If the freestream vector is 10i + 3j m/s and the circulation is -3k m²/s in the above, what is the lift per unit span?
3. If the angle of attack is 5 degrees and the chord is 0.5 in the above, what is the airfoil lift curve slope per radian? Does that make sense?
4. If the lift curve slope of an airfoil is 5.8 per radian, and the aspect ratio is 6, what is the wing lift curve slope?
5. In #4, at an angle of attack of 6 degrees, what is the wing lift coefficient, and what is the induced drag coefficient?
6. If the wing of #5 is at the speed for minimum drag of the airplane, what is the profile drag coefficient of the airplane?
7. Prove your answer to No. 6 by derivation.
8. What is the shape of the wing spanwise lift distribution for best L/D? Why?
9. Look up data on an airfoil. Does the profile drag coefficient in fact vary with angle of attack? How large is this drag coefficient compared to the induced drag coefficient over the range of angles of attack from 0 to stall, for a wing of aspect ratio 8?
Compressible Flow

Speed of sound in a medium

Speed at which ‘infinitesimal disturbances are propagated into an undisturbed medium.

Mach cones and "zones of silence"

In a flow where the velocity is < a (i.e. subsonic flow), disturbances are felt everywhere. When u > a (i.e. supersonic flow), disturbances cannot propagate upstream.

\( \mu \) is the "Mach Angle"
\[
\frac{u \Delta t}{a \Delta t} = M = \frac{1}{\sin \mu} \Rightarrow \mu = \sin^{-1}\left(\frac{1}{M}\right)
\]

Bigger disturbances propagate faster (e.g. shocks). This can be seen by considering the dependence of ‘a’ on local properties.

\[
a = \sqrt{\gamma RT}
\]

If \(T\) is substantially higher downstream of the disturbance, ‘a’ downstream is > ‘a’ upstream. Also,

\[
a^2 = \frac{\partial p}{\partial \rho}
\]

constant entropy
Isentropic Flow Relations

\[
\frac{T_0}{T} = 1 + \frac{r - 1}{2} M^2, \text{ where } \gamma = \frac{c_p}{c_v}
\]

\[
\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{r}{r-1}}
\]

\[
\rho = \frac{P}{RT}, \text{ where } R = \frac{R_{\text{univ}}}{\text{Molecular wt}} = \frac{8313}{28.8} \text{ in S.I. units for air}
\]

\[
c_p = \frac{R \gamma}{r-1}, \text{ specific heat at constant pressure}
\]

\[
c_v = c_p - R, \text{ specific heat at constant volume}
\]

\[
h = c_p T, \text{ enthalpy}
\]
Quasi-1-dimensional flow

Velocity-Area Relation

\[
\frac{dA}{A} = (M^2 - 1) \frac{du}{u}
\]

Thus, if \( M > 1 \) (supersonic),
- \( dA > 0 \) so that \( du > 0 \) (expansion)
- \( dA < 0 \) so that \( du < 0 \) (compression)

and if \( M < 1 \) (subsonic)
- \( dA > 0 \) so that \( du < 0 \) (compression)
- \( dA < 0 \) so that \( du > 0 \) (expansion)
Mass flow rate through a choked throat

\[ m = \rho A U = \frac{P}{RT} (M \sqrt{RT^\gamma})(A) = \left( \frac{P}{P_0} \right) \left( \frac{T_0}{T} \right) \left( \frac{T}{T_0} \right) (M) \left( \frac{\sqrt{R}}{R} \right) \left( A \right) \left( \frac{P_0}{\sqrt{T_0}} \right) \]

choked throat: \( M = 1 \)

\[ m = M (1 + \frac{r-1}{2} M^2) \left( \frac{1 - \frac{r}{r-1}}{2} A\sqrt{R \Lambda} \right) \left( \frac{P_0}{R} \right) \left( \frac{P_0}{\sqrt{T_0}} \right) \]

or,

\[ m \propto \frac{P_0}{\sqrt{T_0}} \]
Normal Shocks

\[ T_0 \]

\[ P_0 \]

\[ T \]

\[ P \]

\[ M_{1,0} \]

\[ u \]
These changes in properties can be calculated using the normal shock relations.

Normal shock relations:

\[ \rho_1 u_1 = \rho_2 u_2 \]

\[ p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \]

\[ T_1 + \frac{u_1^2}{2c_p} = T_2 + \frac{u_2^2}{2c_p} \]

\[ M_2^2 = \frac{M_1^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{r - 1}M_1^2 - 1} \]

\[ \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \]

\[ \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \]
Oblique shocks

All the above hold true for the normal component of velocity. In addition, the tangential component of velocity remains unchanged across the oblique shock.

\[
\frac{\tan(\beta - \theta)}{\tan \beta} = \frac{u_2}{u_1}
\]

\[
M_2 = \frac{M_{2n}}{\sin(\beta - \theta)}
\]
Limits on $\beta$ for an oblique shock

$U > a$ (M>1)

$\mu < \beta < 90$ deg.
Prandtl-Meyer Expansion

Isentropic;

$T_0, P_0$ constant
Isentropic;

$T_0, P_0$ constant

\[
\frac{du}{d\theta} = \frac{-u}{\sqrt{M^2 - 1}} \quad \frac{dp}{d\theta} = \frac{M^2 \gamma}{\sqrt{M^2 - 1}}
\]

At any point, where the Mach # is $M$,

\[
\Rightarrow \alpha_2 - \alpha_1 = -\left(\sqrt{\frac{1 + \gamma}{\gamma - 1}}\tan^{-1}\left(\sqrt{\frac{1 + \gamma}{\gamma - 1}}(M^2 - 1)\right) - \tan^{-1}\left(\sqrt{M^2 - 1}\right)\right)^{\frac{M^2}{k+1}}
\]

The function of $M$ on the RHS above, is tabulated as the Prandtl-Meyer function, in units of degrees.

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a) Calculate M2 for continuous compression turns of 10 degrees and 30 degrees.
b) Calculate M2 from oblique shock relations for turning angles of 10 degrees and 30 degrees.
c) What do you conclude?

Answers: a) 2.527, 1.767. b) 2.51, 1.41 c) the Mach number drop is greater for shocks than for continuous compressions. This is because the continuous compression is “isentropic” (and hence incur no losses in stagnation pressure). Shocks incur a drop in stagnation pressure and a jump in entropy. So the flow has less momentum left. Mach number and velocity both decrease.
Determine $M_2$, $p_2$, $p_{02}$

Answers: 4.319, 1.595 psia, 367.33 psia
Determine $\Phi_1$, $\Phi_4$, $p_2$, $p_3$, $p_04$. The two ramp angles are both 20 degrees. The angles $\Phi_1$ and $\Phi_4$ are measured from the freestream direction.

Answers: 53 deg., 20.2 deg., 28.5 psia, 2.955 psia, 58.45 psia
1. A thin wedge-nosed craft (angle looks pretty small but your i-phone does not come with a protractor) is flying at zero angle of attack at Mach 2.5. Estimate the leading shock angle and justify your estimate.

2. What is the stagnation temperature at a missile nose at Mach 0.8, at 10 km pressure altitude?

3. Calculate the maximum possible turning angle through a Prandtl-Meyer expansion from Mach 3.0.

4. What is the minimum ratio of upstream stagnation pressure to downstream ambient pressure, to get supersonic flow through a nozzle?

5. Calculate the speed of sound at the surface of the planet Xylon, where pressure is 0.1 million Pascals, density is 1 kg/m^3 and the atmosphere is 100% argon. Note: ratio of specific heats for a monatomic gas is 1.67, and for a diatomic gas is 1.4.

6. Develop and validate an Excel spreadsheet calculation for (a) a normal shock and (b) an oblique shock in air. Input is upstream conditions (and turning angle in the case of the oblique shock) and output is downstream conditions. Validation can be done against the tables in the textbook.

7. Develop a calculation for Prandtl-Meyer expansion. Input should be upstream Mach number and turning angle, output should be the downstream Mach number. Validate against the tables in the book.
“Conservation equations” are the reality checks of engineering. They are the laws of physics applied in a form suitable to the problem at hand.
CONSERVATION EQUATIONS

These equations are expressions of the laws of physics, written in forms appropriate for flows.

(i) Mass is neither created nor destroyed: "continuity equation", or "conservation of mass".

(ii) Rate of change of momentum = Net force: "conservation of momentum"

(iii) Energy is conserved, though it may change form: "conservation of energy"

Integral forms for a control volume
\[ \int_{CV} \frac{\partial}{\partial t} \rho dV + \int_{CS} (\rho \vec{u} \cdot \vec{n}) dS = \int_{CV or CS} (\text{whatever}(?) \text{changes to})(dV \text{ or } dS) \]

Mass:
\[ \int_{CV} \frac{\partial}{\partial t} (\rho) dV + \int_{CS} (\rho) (\vec{u} \cdot \vec{n}) dS = 0 \]

Momentum:
\[ \int_{CV} \frac{\partial}{\partial t} (\rho \vec{u}) dV + \int_{CS} (\rho \vec{u}) (\vec{u} \cdot \vec{n}) dS = - \int_{CS} p \vec{n} dS + \int_{CV} \rho \vec{f} dV + F_{\text{shear}} \]

The terms on the rhs are:
I: pressure forces acting normal to the surface, per unit area.
II: body forces per unit mass.
III: shear forces acting parallel to the surface, per unit area.
Energy:

\[ \phi \left( \frac{\partial}{\partial t} \right) \left[ \rho \left( e + \frac{u^2}{2} \right) \right] dV + \phi \left[ \rho \left( e + \frac{u^2}{2} \right) \right] (\vec{u} \cdot \vec{n}) dS = \phi \rho \dot{x} dV + \phi \rho \dot{y} dV \]

Note:

\[ u^2 = |\vec{u}|^2 \]

Energy per unit volume:

\[ \rho \left( e + \frac{u^2}{2} \right) = \text{(mass per unit volume)} \times \text{(Internal energy per unit mass + kinetic energy per unit mass)} \]

In addition to these specific conservation laws, specific equations relating the state variables are needed to solve problems for given kinds of fluids.

To solve very complicated flow problems, the boundary conditions are specified, and all of these equations are solved simultaneously all over the flowfield, for each step in time. This is a task which usually requires fast computers with large memory, because we have to keep track of a large number of variables and perform a large number of calculations.

Long before this became "possible" people figured out more restricted ways of solving specific problems needed to build airplanes. These "smart analytical methods" form the subject of this course.

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Relations Used to Reduce the Conservation Equations to Differential Form

1. Stokes' Theorem

\[ \oint_{C} \vec{u} \cdot d\vec{l} = \oint_{A} \left( \nabla \times \vec{u} \right) \cdot d\vec{A} \]

where \( \vec{u} \) is the vector quantity of interest, \( d\vec{l} \) is the vector along the closed contour of integration \( C \), \( d\vec{A} \) is the unit vector normal to the area enclosed by \( C \).
2. Divergence Theorem

\[ \phi \mathbf{u} \cdot d\mathbf{A} = \phi (\nabla \cdot \mathbf{u}) dV \]

\[ A \quad V \]

3. Gradient Theorem

If \( p \) is a scalar field, then

\[ \phi pd\mathbf{A} = \phi \nabla pdV \]

\[ A \quad V \]

4. \( \nabla \cdot (\rho \mathbf{u}) \equiv \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho \) \quad \text{where} \quad \rho \quad \text{is a scalar and} \quad \mathbf{u} \quad \text{is a vector.} \]
Substantial Derivative

The Eulerian Frame of Reference is the one fixed to the control volume. The Lagrangian frame of reference is the one fixed to a packet of fluid (a fluid element).

The rate of change of any property as seen by the fluid element is:

\[
\frac{D(\cdot)}{Dt} = \frac{\partial}{\partial t}(\cdot) + u \frac{\partial}{\partial x}(\cdot) + v \frac{\partial}{\partial y}(\cdot) + w \frac{\partial}{\partial z}(\cdot)
\]
The substantial derivative is:

\[ \frac{D(?)}{Dt} = \frac{\partial}{\partial t}(?) + (\hat{u} \cdot \nabla)(?) \]

The first term on the rhs is the "local" or "unsteady" term. The second is the "convective" term.

The rate of change \(\frac{D()}{Dt}\) is for two reasons:

1. **Things are changing at the point through which the element is moving (unsteady, local)**

2. **The element is moving into regions with different properties.**
Using the vector identity (4) above, the conservation equations can be re-written:

**Continuity:**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \hat{u}) = 0 \quad \text{or,}
\]

\[
\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \hat{u} + \hat{u} \cdot \nabla \rho = 0
\]

\[
\frac{D \rho}{Dt} + \rho \nabla \cdot \hat{u} = 0
\]

In terms of velocity components, this can be written as a scalar equation:

\[
\frac{D \rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]
Momentum Conservation: Differential Form

\[ \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + \langle F_{viscous} \rangle_x \]

\[ \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y + \langle F_{viscous} \rangle_y \]

\[ \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z + \langle F_{viscous} \rangle_z \]
Knowing the properties of the particular fluid and problem being considered, the body force term and the viscous force term can be expanded. One very useful form is where the viscous stresses are related to the rate of strain of the fluid, through a linear expression. This is valid for "Newtonian Fluids". This is further simplified using the Stokes hypothesis, which permits us to delete the normal-strain terms from the strain terms, leaving only shear-strain terms. The resulting form of the momentum equation is called the Navier-Stokes equation. This is often used as the general starting point to solve problems in fluid mechanics.

**Energy Conservation**

\[ \rho \frac{D(e + \frac{u^2}{2})}{Dt} = \rho \dot{q} - \nabla \cdot (p\vec{u}) + \rho (\vec{f} \cdot \vec{u}) + Q'_{viscous} + W'_{viscous} \]
The Euler equation

If the Reynolds number $\text{Re} \equiv \frac{\rho UD}{\mu} = \text{Inertial Force divided by Viscous Force} >> 1$ in our flow problem, we can neglect the viscous stress terms. Thus the differential form of the momentum equation reduces to:

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho \hat{u})u = -\frac{\partial p}{\partial x} + \rho f_x$$

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho \hat{u})v = -\frac{\partial p}{\partial y} + \rho f_y$$

$$\frac{\partial}{\partial t}(\rho w) + \nabla \cdot (\rho \hat{u})w = -\frac{\partial p}{\partial z} + \rho f_z$$
Here \( u, v, w \) are the Cartesian components along \( x, y, z \) of the vector \( \vec{u} \), and \( f_x, f_y \) and \( f_z \) are components of the body force vector. From the continuity equation,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
\]

and the substantial derivative, we can reduce the momentum equation to:

\[
\rho \frac{Du}{Dt} = -\nabla p + \rho \vec{f}
\]

Euler equation.