Low Speed Aerodynamics

Notes 5: Potential Flow Method
Objective:
Get a method to describe flow velocity fields and relate them to surface shapes consistently.

Strategy:
Describe the flow field as the effect of variations of one quantity.

Vector Identity: Given a scalar function $\Phi$

$$ \nabla \times (\nabla \Phi) \equiv 0 $$

$\Phi$ may be $\Phi(x, y, z, t)$

The "curl" of the gradient of a scalar function is zero.
The velocity potential

We defined the "vorticity" of flow as \( \xi = \nabla \times \vec{U} \)

This is a measure of the rotation of the flow. So \( \nabla \times \vec{U} = 0 \) means "irrotational flow."

Since velocity is a vector, and is a function of \((x, y, z, t)\), this means that if the flow is irrotational, you can define a scalar function \( \Phi(x, y, z, t) \) such that

\[
\vec{U} = \nabla \Phi
\]

\( \Phi(x, y, z, t) \) is called the "velocity potential."

The definition implies:

\[
\begin{align*}
u &= \frac{\partial \Phi}{\partial x} \\
v &= \frac{\partial \Phi}{\partial y} \\
w &= \frac{\partial \Phi}{\partial z}
\end{align*}
\]

The condition for existence of a velocity potential is that the flow should be IRROTATIONAL. The speed of Mach number need not be low.
Incompressible Flow

The continuity equation (conservation of mass) in differential form is

\[ \frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = 0 \]

or

\[ \frac{D\rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]

\[ \frac{D\rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]

Suppose that the changes in density anywhere in this flow are very small, and hence negligible.

\[ \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]

If density is zero, there is no mass, and hence no flow. So the useful solution is

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

“Dilatation is zero”
Incompressible Potential Flow: Laplace Equation

Incompressible: Continuity Equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Irrotational: \( \nabla \times \vec{U} = 0 \) So the velocity can be written as the gradient of a scalar Potential

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}$$

Substituting, the Continuity equation becomes the Laplace Equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{or} \quad \nabla^2 \Phi = 0$$

So, what is the Laplace Equation, for incompressible potential flow?

It is simply the continuity equation!
Same equation describes STEADY or UNSTEADY incompressible potential flow
The lift and pitching moment produced by a thin cambered airfoil can be represented by a vortex sheet placed along the mean camber line of the airfoil. The issue is to find the unknown variation of vortex sheet strength $\gamma(x)$.

The boundary condition to satisfy is that the velocity must be tangential to the vortex sheet at the vortex sheet. Also, that the trailing edge is a stagnation line.
\[
\begin{align*}
\omega_i(x) &= U_i \left( \alpha - \frac{dz}{dx} \right) \\
\nu(\nu(x)) &= \frac{\gamma(x')}{2\pi \langle x, x' \rangle} \\
\mu(x) &= \frac{1}{2\pi} \int_0^1 \frac{\gamma(x') dx'}{x-x'} \\
&= \frac{1}{2\pi U} \int_0^1 \frac{\gamma dx'}{x-x'} = \alpha - \frac{dz(x)}{dx} \\
\cos \theta &= 1 - 2x \quad (\theta \text{ varies from } 0 \text{ to } \pi) \\
\gamma(\theta) &= 2 U \left[ A_0 \cot \frac{\theta}{2} + \sum_{1}^{\infty} A_n \sin n\theta \right]
\end{align*}
\]
\[
C_{m_{1.e.}} - C_{m_{2.e.}} = \frac{\rho U_\infty y}{\rho U_\infty^2/2} = \frac{2y}{U_\infty} = 4 \left[ A_0 \cot \frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]
\]

\[
1 = \int_0^1 \rho U_\infty y \, dx \quad \text{or} \quad C_1 = 2\pi \left( A_0 + \frac{A_1}{2} \right)
\]

\[
m_{1.e.} = \int_0^1 \rho U_\infty y \, x \, dx \quad \text{or} \quad C_m = \frac{\pi}{2} \left( A_0 + A_1 - \frac{A_2}{2} \right) = \frac{C_1}{4} \cdot \frac{\pi}{4} (A_1 \cdot A_2)
\]
If the airfoil is represented by a vortex sheet along the camber line, then the camber line is a streamline of the flow.

The velocity induced at a point \( x \) by the entire vortex sheet is

\[
\vec{w}(x) = -\frac{c}{2\pi} \int_0^c \frac{\gamma(\xi)}{(x - \xi)} d\xi
\]

where \( \xi \) is the distance along the chord line.

The thin airfoil assumption leads to the argument that \( \vec{w'}(s) \approx \vec{w}(x) \)

where \( \vec{w} \) is the component of velocity normal to the vortex sheet at the camber line.

**Boundary Condition**

\[
\vec{U}_\infty \cdot \vec{n} + w'(s) = 0
\]

\[
\vec{U}_\infty \cdot \vec{n} = \left| \vec{U}_\infty \right| \sin(\alpha - \tan^{-1}\left(\frac{d}{dx}(z)\right)) \approx \left| \vec{U}_\infty \right| \chi
\]

Note from Figure:

\[
\frac{1}{2\pi} \int_0^c \frac{\gamma(\xi)}{(x - \xi)} d\xi = \left| \vec{U}_\infty \right| \left( \alpha - \frac{dz}{dx} \right)
\]

for small angle of attack and small surface slope.
This is called the \textbf{Fundamental Equation of Thin Airfoil Theory}. It says:

\textbf{Camber line is a streamline of the flow}.

\textbf{Results for a Symmetric Airfoil}

Symmetric airfoil: the camber line is straight.

Use the transformation: \[ \xi = \frac{c}{2}(1 - \cos \theta) \]

\[ x = \frac{c}{2}(1 - \cos \theta_0) \]

\[ \frac{1}{2\pi} \int_{0}^{c} \frac{\gamma(\xi)}{(x - \xi)} d\xi - U_{\infty} \alpha \]

\[ \gamma(\theta) = 2\alpha U_{\infty} \frac{(1 + \cos \theta)}{\sin \theta} \]

\[ \frac{1}{2\pi} \int_{0}^{c} \frac{\gamma(\theta) \sin \theta}{(\cos \theta - \cos \theta_0)} d\theta = U_{\infty} \alpha \]
\[ \gamma(\theta) = 2\alpha U_\infty \frac{(1 + \cos \theta)}{\sin \theta} \]

Consider some features of this function:

When you go close to the leading edge, the value goes shooting up towards infinity. As you go towards the trailing edge, the value goes to zero.

Total circulation around the airfoil is:

\[ \Gamma = \int_0^\infty \gamma(\xi) d\xi \qquad \Gamma = \int_0^\pi \gamma(\theta) \sin \theta d\theta \]

Therefore,

\[ \Gamma = \pi \alpha c U_\infty \]

\[ L' = \rho U_\infty \Gamma = \rho \pi \alpha c U_\infty^2 \]

\[ L' = \frac{1}{2} \rho U_\infty^2 c_l \]

\[ c_l = 2\pi \alpha \]

\[ \frac{dc_l}{d\alpha} = 2\pi \]
Pitching moment about the leading edge

\[ M'_{LE} = -\int_0^c \xi dL = -\rho U_\infty \int_0^c \xi \gamma(\xi) d\xi \]

This reduces to:

\[ M'_{LE} = -q_\infty c \frac{2\pi \alpha}{2} \]

The moment coefficient is:

\[ c_{mLE} = \frac{M'_{LE}}{q_\infty c^2} = \frac{-\pi \alpha}{2} \]

\[ c_{mLE} = \frac{-c_l}{4} \]
negative value means that the pitching moment about the leading edge is nose-down. No wonder. T

The location of the center of pressure can be found as follows: The center of pressure is the point t

\[ x_{cp} = \frac{-M^L_{LE}}{L'} = -\frac{c(c_{p_{LE}})}{c_l} = \frac{c}{4} \]
Interesting: The resultant lift acts through the quarter-chord point. In other words, the moment due to the lift is obtained by multiplying the bound circulation by some constants. The bound circulation is of course determined from the vortex sheet.

Why the vortex sheet? The vortex sheet represents the difference between upper surface velocity and lower surface velocity. The vortex sheet is generated by the separation of the flow, and its strength determines the lift.

So, as we trace back through the derivation, we see that the variation of lift along the chord was determined from the variation of the bound circulation along the chord. Most of the lift on a symmetric airfoil in incompressible flow comes from the suction peak occurring pretty close to the leading edge.

This is indeed borne out by measurements: the suction peak occurs pretty close to the leading edge.
Now since the center of pressure is at \( c/4 \) regardless of angle of attack (as long as the angle of attack is small, like below 10 degrees or so), the pitching moment about this point (\( c/4 \), or \( \text{quarter-chord} \)) must be zero, regardless of angle of attack.

The Aerodynamic Center

The aerodynamic center of an airfoil is the point located such that the pitching moment about that point is independent of angle of attack. The moment about \( c/4 \) is zero regardless of angle of attack, so this qualifies as the aerodynamic center.

Thus, for a symmetric airfoil, the center of pressure and the aerodynamic center are both at the quarter-chord in low-speed flow.
If the airfoil is cambered, the center of pressure moves back from the quarter-chord. The aerodynamic center is still at the quarter-chord point. This can be seen as follows: We can consider the lift on the cambered airfoil to come from two things: the camber, and the angle of attack. The lift due to camber is independent of angle of attack, and acts through some point on the airfoil depending on the camber line shape. This is the lift when the angle of attack is zero. The part due to angle of attack still acts through the center of pressure. So if we take moment about the quarter-chord, we will get zero from the angle-of-attack part, regardless of angle of attack. We will get a constant value of moment due to the camber part, but this again will not change with angle of attack. So the pitching moment through the quarter-chord is still independent of angle of attack (though not zero). Thus the quarter-chord point is the aerodynamic center even for an cambered airfoil.

Now as speed increases into the regime where the flow density changes appreciably due to changes in velocity (the compressible regime), the center of pressure moves back over the airfoil, even for symmetric airfoils.

In supersonic flow, the center of pressure of a thin symmetric airfoil is at the midchord. This can be seen from supersonic thin airfoil theory.
Results for a Cambered Airfoil
(the derivation can be found in several textbooks, but is simply substitution in the relations derived before).

Assume that the camber line is given by \( z = z(x) \), with \( z = 0 \) at \( x = 0 \) and \( x = c \).

The lift coefficient is:

\[
c_l = 2\pi \left[ \alpha + \frac{1}{\pi} \int_0^\pi \frac{d}{dx} (\cos \theta - 1) d\theta \right]
\]

Obviously, this consists of two parts: the first part is that due to angle of attack, and second due to camber. When the angle of attack for zero lift is thus:

\[
\alpha_l = 0 = -\frac{1}{\pi} \int_0^\pi \frac{d}{dx} (\cos \theta - 1) d\theta
\]

We have to leave these things as integrals because the precise expression will depend on the shape of the camber line for the given airfoil, i.e., the function \( \frac{dz}{dx} \).

The coefficients \( A_1 \) and \( A_2 \) are seen from the following integrals:

\[
c_{mLE} = \frac{-c_l}{4} = \frac{\pi}{4} (A_1 - A_2)
\]

\[
A_0 = \alpha \frac{1}{\pi} \int_0^\pi \frac{d}{dx} d\theta
\]
pitching moment about the quarter-chord is:

\[ A_n = \frac{2}{\pi} \int dx \cos (n \theta_0) d\theta_0 \]

The location of the center of pressure is now:

\[ x_{cp} = \frac{c}{4} \left( 1 + \frac{\pi}{c} (A_1 - A_2) \right) \]
Why the Laplace Equation is Useful for Incompressible Flow

For incompressible flow conditions, velocity is not large enough to cause appreciable density changes, so density is known - constant. Thus the unknowns are velocity and pressure. We need two equations. The Continuity equation is enough to solve for velocity as a function of time and space.

• Pressure can be obtained from the Bernoulli equation, which comes from Momentum Conservation.
• If the flow is irrotational as well, we can define a potential, and reduce the continuity equation to the form of the Laplace equation. \[ \nabla^2 \Phi = 0 \]

\[
\vec{U} = \nabla \Phi \quad u = \frac{\partial \Phi}{\partial x} \quad v = \frac{\partial \Phi}{\partial y} \quad w = \frac{\partial \Phi}{\partial z}
\]

The flow around a complete aircraft is routinely calculated from this.

**Problem:** If the flow is irrotational, how can there be lift?

**Solution:** We will introduce rotation, and declare the source of rotation to be “outside” the irrotational flow. This strange logic works, as we will see.
Solving The Laplace Equation

People have found several neat solutions to this equation. They have also found that it is a LINEAR equation, so that you can add one solution to another, and get a third solution!

So we get the solution to flow around an airplane, as the sum of, say, 3000 little solutions to the Laplace equation, distributed all over the flow and airplane surface.

The difficulty is in calculating the values of these 3000 (or whatever our computer can handle – so that their sum “obeys the boundary conditions” that define our flow problem.

Figure 5: Grid topology for MGAERO
Figure 6: MGAERO-computed surface pressures on the Dash 8-400 showing the aft fuselage ventral strakes and fin-cap fairing

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Solving The Laplace Equation: Superposition

If each of the following: \( \Phi_1, \Phi_2, \Phi_3, \Phi_4 \) is a solution of the Laplace equation,

then \( A\Phi_1 + B\Phi_2 + C\Phi_3 + D\Phi_4 \) is also a solution, where \( A, B, C, D \) etc. are constants.

Thus, the solution for a complex problem can be expressed as the sum of solutions of several simpler problems.

We will use these building blocks to analyze several flows:

1. Uniform flow + source:
2. Uniform flow + source + sink.
3. As source and sink come close together, you get a doublet.
4. Uniform flow + vortex. This looks like flow over a spinning cylinder.
We see that (1), (2) and (3) are all symmetric about the freestream direction. (4) is not.
Boundary Conditions
This is what specifies the details of the problem in precise mathematical terms.

(i) Along the surface of the airfoil, the flow must be tangential to the surface:
Component of velocity normal to the surface is zero. i.e.,

\[ \nabla \Phi \cdot \vec{n} = 0 \quad \vec{U} \cdot \vec{n} = 0 \]

or,

\[ \frac{\partial \Phi}{\partial n} = 0 \]

Thus, all the flow near the surface must be tangential to the surface. In other words, the streamline closest to the surface must be parallel to the surface.

Thus another way to express condition (1) for 2-d problems is to say that “the surface is a streamline”
The far-field boundary condition

(ii) The disturbance due to an object must die away as you go far from the surface (any solution which says otherwise is physically unrealistic).

Thus,

\[ A_s \]

\[ x, y, z \to \pm \infty \]

\[ \vec{U} \to \vec{U}_\infty \]

\[ |\vec{U} - \vec{U}_\infty| \to 0 \]

Example:

Suppose the general solution to a differential equation is

\[ \phi = Ae^x + Be^{-x} \]

What can you say about \( A \) or \( B \) from the above condition?
Elementary Solutions of the Laplace Equation

(i) Uniform flow:

\[ u = u_\infty = \frac{\partial \phi}{\partial x}; \quad v = \frac{\partial \phi}{\partial y} = 0 \]

\[ \phi = u_\infty x + f(y), \quad \phi = \text{const} + g(x). \]

\[ \phi = u_\infty x + \text{const.} \quad \Rightarrow \quad \phi = u_\infty x \]

\[ \Rightarrow f(y) = \text{const.}, \quad g(x) = u_\infty x. \]
(ii) Source or sink (note: a sink is simply a negative source).

Except at the origin,

At the origin, there is mass flow being added or subtracted ["singular point"]. [There's flow out of the plane if 2-D.]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ u_r = \frac{c}{r} \]

\[ \frac{\partial \phi}{\partial r} = u_r = \frac{\lambda}{2\pi r} \quad \Rightarrow \quad \phi = \frac{\lambda}{2\pi} \ln r + f(\theta) \]

\[ \frac{1}{r} \frac{\partial}{\partial \theta} = u_\theta = 0 \quad \Rightarrow \quad \phi = \text{const} + f(r) \]

\[ u_\phi = 0 \]

\[ c = \frac{\lambda}{2\pi} \frac{m^2}{s} \]

\[ \phi = \frac{\lambda}{2\pi} \ln r \]

where \( \lambda \) is the source strength:

At the origin, there is mass flow being added or subtracted ["singular point"].

There's flow out of the plane if 2-D.

\( \lambda \) is a constant, related to the volume flow from/to the source/sink.
(iii) Doublet

\[ \phi = \frac{\kappa \cos \theta}{2\pi r} \]
(iv) Vortex

Let us take the "circulation" around the vortex at radius \( r \).

\[ u_\phi = \text{const.} \Rightarrow u_\phi = \frac{-\Gamma}{2\pi r} \]

Note that if you take the circulation at any radius, you'll get the same value of \( \Gamma \).

The only vorticity is concentrated at the center; the flow is irrotational everywhere except at the center.

What is special about the vortex? Let's try computing the circulation for the first 3: we get zero.

\[ r \equiv -\int u_\cdot \, ds = -u_\phi (2 \pi r) \]

\[ \Gamma = \oint u_\cdot \, ds = 0 \]

\[ \frac{\partial \phi}{\partial r} = u_r = 0 \]

\[ \phi = -\frac{\Gamma}{2\pi r} \theta \]

\[ \frac{1}{r} \frac{\partial \phi}{\partial \theta} = u_\theta = -\frac{\Gamma}{2\pi r} \theta \]
The Biot-Savart Law

Consider a segment of a vortex filament as shown, with strength $\Gamma$

Velocity induced at point $P$ by the segment $dl$ is

$$d\vec{u} = \frac{\Gamma}{4\pi} \frac{dl \times \vec{r}}{|\vec{r}|^3}$$

when $\Theta$ is the angle between $\vec{r}$ and $\Gamma$

$d\vec{e} \times \vec{r}$ is $|d\vec{e}| |\vec{r}| \sin \Theta$

Thus, velocity induced at $P$ by an infinite, straight vortex filament is

$$\Rightarrow \vec{u} = \frac{\Gamma}{2\pi h}$$

where $h$ is the distance $\perp \vec{r}$ to the vortex sheet.

Thus, the induced velocity drops off as $1/h$ and $h$ increases.
Note: This is a general result for potential fields, taken by analogy from the result for the magnetic field induced by a segment $\vec{a}\vec{E}$ of a conductor carrying current $I$, in a medium of permeability $\mu$:

$$d\vec{B} = \frac{\mu I}{4\pi} \frac{\vec{a} \times \vec{r}}{|r|^3}$$
The Vortex Sheet

A vortex sheet is a continuous sheet of vortices.

It is used to represent a "shear layer", across which the velocity changes, as shown below.

The velocity induced at P by an infinitesimal portion $d\ell$ of the sheet is

$$ u = \frac{\Gamma}{4\pi} \int_{\infty}^{\infty} \frac{\sin \theta}{r^2} d\ell $$

$$ \hat{u} = \frac{\Gamma}{4\pi} \int_{\infty}^{\infty} \frac{\vec{d\ell} \times \hat{r}}{r^2 \rho^2} $$

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Define \( \gamma \) as the strength of the vortex sheet per unit length of the vortex sheet. Thus, strength of an infinitesimal portion \( \delta \gamma \) of the sheet is \( \delta \gamma \).
Define $\Gamma$ as the strength of the vortex sheet per unit length of the vortex sheet.
Thus, strength of an infinitesimal portion of the sheet is $d\Gamma$.

Velocity induced at $P$ by this segment is $v$.
Velocity potential at $P$ due to vortex sheet segment is $\Phi$ where $\theta$ is the angle of $r$ from your axis reference.
\[ \Gamma = \int_{a}^{b} \gamma dl - \int \vec{u} \cdot dl \]

\[ \Gamma = -\left( v_2 \hat{n}dn - u_1 dl - \nu_1 \hat{n}dn + u_2 dl \right) \]

Circulation around the segment \( a > b \) is

Circulation is also

As \( \hat{n} > 0 \),

Local jump in tangential vel. across vortex sheet = local sheet strength

\[ \gamma = u_1 - u_2 \]

\[ = -\left[ (u_2 - u_1) dl + (v_2 - \nu_1) dn \right] \]
Example 1 Question: How can we put in a set of simple flows so that the streamlines look the same as if there were an airfoil present? Overall, what happens is
Close up, we see other effects:
Example 2
Quiz 2

No. of Students

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