Low Speed Aerodynamics

Notes 8: Viscous Flow
Shear Stress vs. Rate-of-Strain Relations

\[ \sigma_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

The direction is indicated by the second subscript. The units of stress are Force/Area, such as Newtons/m², psi, or psf.

\[ \sigma_{yz} = \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \]

\[ \sigma_{zx} = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \]

These can be written in compact form. Here \( u \) is used to represent any of the velocity components, and \( x \) is used to represent any of the spatial coordinates. \( i \) and \( j \) representing \( x,y,z \) in turn.

\[ \sigma_{ij} = \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \]
Normal stresses due to viscosity

\[
\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

Stokes’ Hypothesis:

\[
\lambda = -\frac{2}{3} \mu
\]

So the normal stresses due to viscosity add up to zero.
2D Navier-Stokes Equations For Incompressible Flow

Cons. of mass:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

Cons. of x-momentum:

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial \Phi}{\partial x} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Cons. of y-momentum:

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial \Phi}{\partial y} = \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]
Exact Solution to the Steady 2D Navier-Stokes Equations For Incompressible Flow

Plane Parallel Steady Flow Between Two Infinite Plates: Couette Flow

From the problem, there cannot be any change in u along x: So $d(u)/dx$ is zero. Also, $v = 0$. So $u$ is $u(y)$

$$1 \frac{dp}{\rho \ dx} = \nu \frac{d^2 u}{dy^2}$$

$$u(y) = \frac{1}{\mu} \frac{dp}{\ dx} \frac{y^2}{2} + Ay + B$$
Case 1: Flow between two plates, driven by plate motion

Boundary conditions: \( u=v=0 \) at \( y=0 \)
\[ u = U \text{ and } v=0 \text{ at } y=h. \]

\[
\begin{align*}
u(y) &= \frac{1}{2\mu} \frac{dp}{dx} \left(y - h\right) + \frac{Uy}{h} \\
\text{If } \frac{dp}{dx}=0, \quad u(y) &= \frac{Uy}{h} \\
\text{Linear velocity profile}
\end{align*}
\]

Shear stress
\[
\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{U}{h}
\]

Skin Friction Coefficient
\[
C_f = \frac{\tau_{xy}}{\frac{1}{2} \rho U^2} = \frac{2\mu}{\rho h U} = \frac{2}{\text{Re}}
\]

Re = Reynolds number based on \( U \) and \( h = \frac{\rho U h}{\mu} \).
Case 1: Flow between two stationary plates, driven by pressure gradient

\[ u(y) = \frac{y}{2\mu} \frac{dp}{dx} (y - h) \]

( Note: Sign of \( \frac{dp}{dx} \) must be opposite to that of \( u \) )
Hagen-Poiseuille Flow: Steady axisymmetric, fully-developed incompressible laminar flow through a constant-diameter pipe

\[
\frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial (r u)}{\partial r} = 0
\]

\[
u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial z} = \nu \left[ \frac{1}{r} \frac{\partial (r \frac{\partial u}{\partial r})}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right]
\]

\[
u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = \nu \left[ \frac{1}{r} \frac{\partial (r \frac{\partial v}{\partial r})}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right]
\]

u-momentum equation reduces to

Integrating,

\[
\frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{v}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right)
\]

\[
\frac{du}{dr} = \frac{1}{2 \mu} \frac{dp}{dz} r + \frac{C}{r}
\]

For flow to remain finite at axis \((r=0)\), \(C=0\). Also, \(u = 0\) at the wall, \(r=R\).

\[u(r) = -\frac{1}{4 \mu} \frac{dp}{dz} \left( R^4 - r^4 \right) \quad \text{Parabolic Velocity Profile}\]
Boundary Layer Equations

Newton’s 2nd Law

Conservation Equations in integral form

Convert to differential form

Simplify by use of OM analysis
Boundary Layer Approximations

At very high Reynolds number (~100,000+) based on distance along the surface in the freestream direction,

1. The spatial rate at which properties change across a boundary layer is very large, compared to the rate at which things change along the flow direction. In other words, derivatives with respect to \( y \) are much higher than derivatives with respect to \( x \).
   \[
   \frac{d(\ )}{dy} \gg \frac{d(\ )}{dx}
   \]

2. Order of magnitude analysis shows that the v-momentum equation reduces to

   In other words, static pressure is constant across a boundary layer.
   \[
   \frac{\partial p}{\partial y} \approx 0
   \]
Boundary Layer Equations

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{dp}{dx} = v \frac{\partial^2 u}{\partial y^2} \]

\[ \frac{\partial p}{\partial y} \approx 0 \]

a) The variation of pressure across the boundary layer is small. Thus, we can assume that pressure varies only with respect to \( x \). The pressure field may be computed at the edge of the boundary layer from inviscid flow theories, without any knowledge of the boundary layer characteristics.

b) The \( v \)-component of velocity within the boundary layer is of the order of \( (U_\infty \delta/c) \) where \( \delta \) is the boundary layer thickness.

c) The boundary layer thickness \( \delta \) is very small, and for laminar flows it varies inversely as the square root of Reynolds number \( \rho U_\infty c/\mu \).

d) We will assume the airfoil surface to be flat, and use a Cartesian coordinate system, neglecting surface curvature. This is because the boundary layer thickness \( \delta \) is very small compared to the airfoil surface radius of curvature.
For high Reynolds number flows, our order of magnitude analysis says that the $u$-momentum equation may be approximated as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$v$-momentum equation

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\mu v \partial v}{\partial y}$$

becomes

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} \approx 0$$

$$\frac{\partial p}{\partial y} \approx 0$$
Boundary Layer Features

\[ \delta(x) \]

Du/dy at the wall

Velocity profile

Copyright 2007 N. Komrath. Other rights may be specified with individual items. All rights reserved.
Boundary Layer Parameters

1. Boundary Layer Thickness $\delta$:
   Defined as the y- location where $u/ue$ reaches 0.99%,
   that is the u- velocity becomes 99% of the edge velocity.

2. Displacement Thickness $\delta^*$:
   This is a measure of the outward displacement of the streamlines from the solid surface as a
   result of the reduced u- velocity within the boundary layer. This quantity is defined as
   \[
   \delta^* = \int_0^\infty \left[ 1 - \frac{\rho u}{\rho_e u_e} \right] dy
   \]

3. Momentum Thickness $\theta$:
   This is a measure of the momentum loss within the boundary layer as a result of the reduced
   velocities within the boundary layer.
   \[
   \theta = \int_0^\infty \frac{\rho u}{\rho_e u_e} \left[ 1 - \frac{u}{u_e} \right] dy
   \]
4. Shape Factor $H$ : This quantity is defined as the ratio $\delta^*/q$.

For laminar flows $H$ varies between 2 and 3. It is 3.7 near separation point. Thus excessively large values of $H$ (above 3) indicate that the boundary layer is about to separate. In turbulent flows, $H$ varies between 1.5 and 2.
Wall Shear Stress and Drag

Surface Shear Stress:
The shear stress at the wall can be found from definition of shear stress

\[ \tau_{wall} = \tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{wall} \]

Skin friction Coefficient \( cf \):

\[ cf = \frac{\tau_w}{\left( \frac{1}{2} \rho_e u_e^2 \right)} \]

Skin Friction Drag, \( D \): Shear stress may be numerically integrated over the entire solid surface to give the skin friction drag force along the x-axis:

\[ D = \int \tau_w \, dx \]

Entire Surface

Skin Friction Drag Coefficient \( Cd \): The drag force is usually non-dimensionalized by the freestream dynamic pressure times the chord of the airfoil \( c \), giving the skin friction drag coefficient along the x-axis, \( Cd \).

\[ Cd = \frac{D}{\left( \frac{1}{2} \rho_\infty u_\infty^2 c \right)} \]
Thwaites' Integral method for Laminar Incompressible Boundary Layers

This is an empirical method based on the observation that most laminar boundary layers obey the following relationship (Ref: Thawites, B., Incompressible Aerodynamics, Clarendon Press, Oxford, 1960).

\[
\frac{u_e}{\nu} \frac{d(\theta^2)}{dx} = A - B \frac{\theta^2}{\nu} \frac{du_e}{dx}
\]  

(1)

Thwaites recommends A = 0.45 and B = 6 as the best empirical fit.
The above equation may be analytically integrated yielding

$$\theta^2 = \frac{0.45}{u_e^6} \int_0^x u_e^5 \, dx + \left[ \theta^2 \right]_{x=0}^{x=0} \frac{u_e^6}{u_e^6}$$

For blunt bodies such as airfoils, the edge velocity $u_e$ is zero at $x=0$, the stagnation point. For sharp nosed geometries such as a flat plate, the momentum thickness $q$ is zero at the leading edge. In these cases, the term in the square bracket vanishes.

The integral may be evaluated, at least numerically when $u_e$ is known.
After $\theta$ is found, the following relations are used to compute the shape factor $H$ and the shear stress at the wall $\tau_w$.

For $0 \leq \lambda \leq 0.1$

$$H = 2.61 - 3.75\lambda + 5.24\lambda^2$$

For $-0.1 \leq \lambda \leq 0$

$$H = 2.472 + \frac{0.0147}{0.107 + \lambda}$$

where

$$\lambda = \frac{\theta^2}{\nu} \frac{du_e}{dx}$$

$$\tau_{wall} = \frac{\mu u_e}{\theta} (\lambda + 0.09)^{0.62}$$

Despite the empiricism in the above formulas, Thwaites' integral method is considered to be the best of integral boundary layer methods.
Consider the 2-D unsteady, incompressible viscous flow past an airfoil at a sufficiently high Reynolds number

\[ \text{Re} = \frac{\rho U_\infty c}{\mu} > 300,000 \]

We find that the boundary layer over both the upper and lower surfaces may be divided into three regions. They are (1) laminar region, (2) transitional region, and (3) turbulent region.

Transition Strip: rough strip taped to airfoil close to max thickness location, to trigger disturbances that cause quick transition to turbulence.
The turbulent region has unsteady flow, although the fluctuations are small and occur about some "steady" mean flow levels. 
The velocity profile is fuller than in the laminar flow case. 
Skin friction is higher than in laminar flow.
## Incompressible Boundary Layer on Flat Plate at zero angle of attack: Laminar vs. Turbulent

<table>
<thead>
<tr>
<th>Laminar</th>
<th>Turbulent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = \frac{5.0x}{\text{Re}^{0.5}} )</td>
<td>( \delta = \frac{0.37x}{\text{Re}^{0.2}} )</td>
</tr>
<tr>
<td>( \theta = \frac{0.664x}{\text{Re}^{0.5}} )</td>
<td>( \theta = \frac{0.664}{\text{Re}^{0.5}} )</td>
</tr>
<tr>
<td>( cf = \frac{1.328}{\text{Re}_c^{0.5}} )</td>
<td>( cf = \frac{0.074}{\text{Re}_c^{0.2}} )</td>
</tr>
</tbody>
</table>
Incompressible Turbulent Skin Friction Correlation At Large Reynolds Numbers

Shultz / Grunow equation, modified by Kulfan, Boeing Co..

\[ C_{f_i} = 0.295 \cdot (\log(Re_x))^{-2.45} \quad 10^6 < Re_x < 10^9 \]
Comparison of Incompressible Local Skin Friction Predictions

\[
\frac{1}{\sqrt{C_{fi}}} = 4.15 \times \log (R_{e x} \times C_{fi}) + 1.7
\]

\[
C_{fi} = 0.295 \times (\log (R_{ex}))^{-2.45}
\]

Courtesy, Dr. R. Kulfan, BOEING Co

Copyright 2007 N. Komerath. Other rights may be specified with individual items. All rights reserved.
Airfoil Drag vs. Angle of Attack

Note: When viscous drag and/or flow separation are present, airfoils do have drag that changes with angle of attack. In other words, $C_{D0}$ of an aircraft is not completely independent of angle of attack, over a large range of angle of attack. This has nothing to do with tip vortices etc. So when we must use an averaged value for $C_{D0}$ when we use lifting line theory and say that for a whole aircraft (note nomenclature)

$$C_D = C_{D0} + \frac{C_L^2}{\pi (AR)e}$$

The more usual and general way, is to represent the drag as

$$C_D = C_{D0} + KC_L^2$$

$K$ will now be $> \frac{1}{\pi (AR)e}$

Copyright 2007 N. Komerath. Other rights may be specified with individual items. All rights reserved.
Aircraft Drag Polar

$C_L_{max}$

$C_L$

$C_D$

$C_D_{0}$

Best L/D? Can you prove it?

Copyright 2007 N. Komirath. Other rights may be specified with individual items. All rights reserved.
A positive pressure gradient $dp/dx$ would cause the boundary layer velocity profile to become “unhealthy” – develop an inflexion point.

Beyond that, 3 things can happen:

1. Laminar separation
2. Transition to turbulence.
3. Turbulent separation

Turbulent boundary layer has high velocities close to surface, large $du/dy$ – high skin friction drag (bad). However, very stable profile _delays separation (good!)_
Separation

Stagnation

3-D Separation line

\[ \frac{\partial U}{\partial y} \bigg|_{wall} = 0 \]
Separation Control: Surface Blowing

Cavities
Vortex Interactions & Ground Effect

Technique: Think of the velocity induced at the axis of each vortex by every other vortex. A solid wall is treated as an “Image plane” with an imaginary mirror image of the vortex on the other side.
Rotor Wake Evolution in Axial Flight

Vortex roll-up

Vortex pairing

7' X 10' section

UH-1H model rotor

30' X 31' settling chamber

Copyright 2007 N. Komarath. Other rights may be specified with individual items. All rights reserved.

Vortex Pairing Sequence